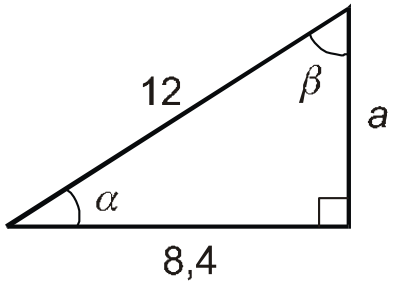
601

Olkoon tuntematon kateetti a ja tuntemattomat kulmat α ja β
Ratkaistaan kulmat.



$$\cos\alpha = \frac{8,4}{12}$$

$$\alpha = \cos^{-1}\left(\frac{8,4}{12}\right)$$

$$\alpha = 45,5729...^\circ$$

$$\alpha \approx 46^\circ$$

$$\beta = 90^\circ - \alpha$$

$$= 44,4270...^\circ \approx 44^\circ$$

Ratkaistaan kateetti a.

$$a^2 + 8,4^2 = 12^2$$

$$a^2 = 12^2 - 8,4^2$$

$$a^2 = 73,44$$

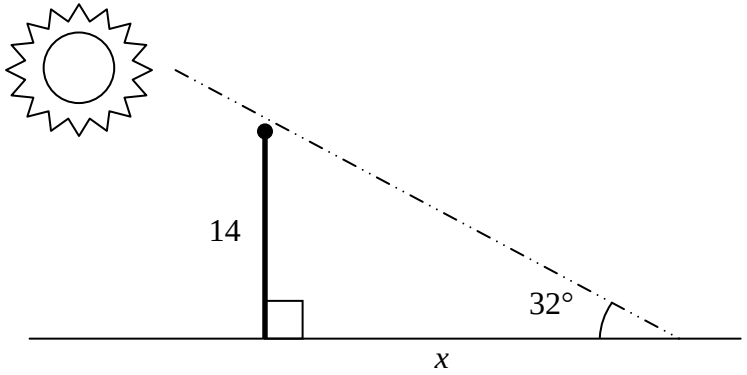
$$a = {}_{(}\pm_{)}\sqrt{73,44}$$

$$a = 8,5697...$$

$$a \approx 8,6$$

Vastaus: Kateetti on 8,6 ja kulmat 46° ja 44°

602



Ratkaistaan varjon pituus x

$$\tan 32^\circ = \frac{14}{x}$$

$$x\tan 32^\circ = 14$$

$$x = \frac{14}{\tan 32^\circ} = 22,404... \approx 22 \text{ (m)}$$

Vastaus Varjo on 22 m pitkä.

603

a) Ratkaistaan x yhtälöstä

$$\tan 52^\circ = \frac{x}{45}$$

$$x = 45 \cdot \tan 52^\circ$$

Leijan etäisyys maanpinnasta on

$$x + 1{,}6 = 45 \cdot \tan 52^\circ + 1{,}6 = 59{,}19... \approx 59$$

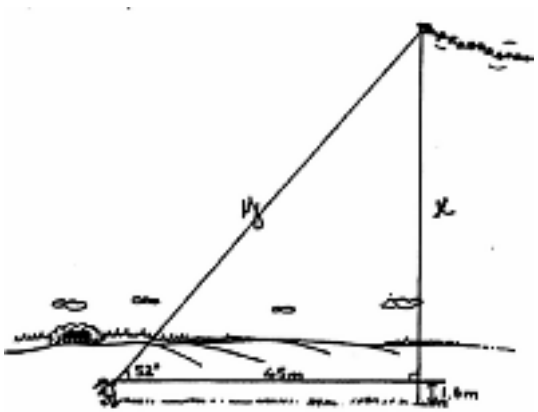
b) Leijan etäisyys Laurista on y.
Ratkaistaan yhtälö

$$\cos 52^\circ = \frac{45}{y}$$

$$y \cdot \cos 52^\circ = 45$$

$$y = \frac{45}{\cos 52^\circ}$$

$$y = 73{,}09... \approx 73$$



Vastaus a) Leijan etäisyys maanpinnasta on 59 m
b) Leijan etäisyys Laurista on 73 m

604

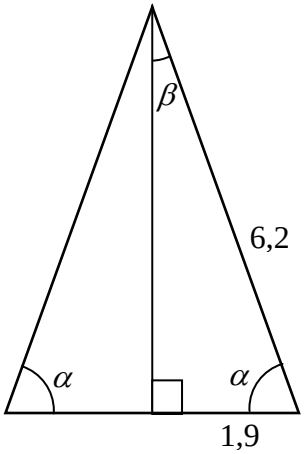
Merkitään tasakylkisen kolmion korkeutta h:lla ja kantakulmia α:lla ja huippukulman puolikasta β:llä.
Ratkaistaan kantakulma.

$$\cos\alpha = \frac{1{,}9}{6{,}2}$$

$$\alpha = \cos^{-1}\left(\frac{1{,}9}{6{,}2}\right)$$

$$\alpha = 72{,}1545...^\circ$$

$$\alpha \approx 72^\circ$$



Kulmaksi β saadaan siten

$$\beta = 90^\circ - \alpha = 17{,}8455...^\circ$$

ja kolmion huippukulmaksi

$$2\beta = 35{,}6910...^\circ \approx 36^\circ$$

Vastaus kantakulmat ovat 72° ja huippukulma on 36°

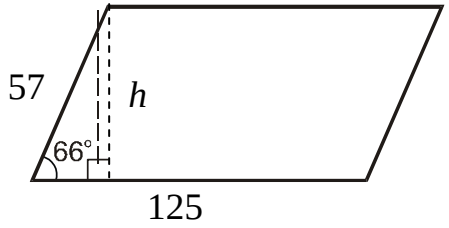
605

Ratkaistaan korkeus h (m).

$$\sin 66^\circ = \frac{h}{57}$$

$$h = 57 \cdot \sin 66^\circ$$

$$h = 52{,}072...$$



Pinta-ala on siten

$$A = 12{,}5 \cdot 52{,}072... \qquad \left| A_{\text{suunnikas}} = \text{kanta} \cdot \text{korkeus} \right.$$

$$= 6509{,}011...$$

$$\approx 6500$$

$$A = 6500 \text{ m}^2 = 65 \text{ a}$$

Vastaus 65 aaria

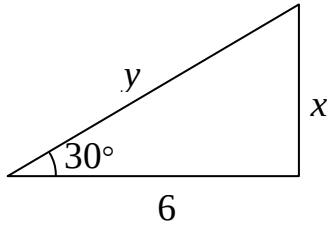
606

Toinen kateetti on

$$\tan 30^\circ = \frac{x}{6}$$

$$x = 6 \cdot \tan 30^\circ$$

$$x = 6 \cdot \frac{1}{\sqrt{3}} = \overset{\sqrt{3}}{} \frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$$



Hypotenuusa on

$$\cos 30^\circ = \frac{6}{y}$$

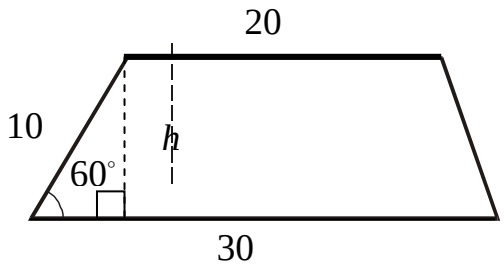
$$y = \frac{6}{\cos 30^\circ}$$

$$y = \frac{6}{\frac{\sqrt{3}}{2}} = \overset{\sqrt{3}}{} \frac{12}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$$

Vastaus Toinen kateetti on $2\sqrt{3}$ ja hypotenuusa $4\sqrt{3}$

607

Ratkaistaan korkeus h.



$$\sin 60^\circ = \frac{h}{10}$$

$$h = 10\sin 60^\circ$$

$$h = 10 \cdot \frac{\sqrt{3}}{2}$$

$$h = 5\sqrt{3}$$

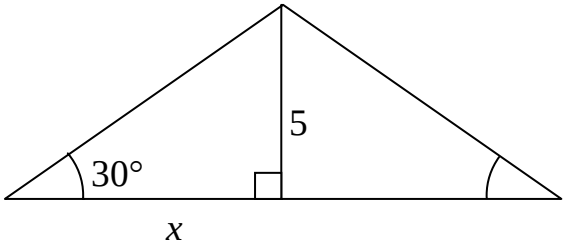
Ala on

$$A = \frac{30+20}{2} \cdot 5\sqrt{3} = 125\sqrt{3}$$

Vastaus $125\sqrt{3}$

608

Merkitään kannan puolikasta x:llä.



Ratkaistaan x.

$$\tan 30^\circ = \frac{5}{x}$$

$$x = \frac{5}{\tan 30^\circ}$$

$$x = \frac{5}{\frac{1}{\sqrt{3}}} = 5\sqrt{3}$$

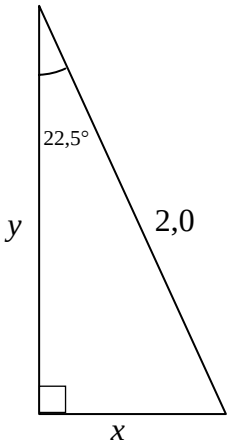
Kanta on siten $2x = 10\sqrt{3}$

Kolmion pinta-ala on

$$A = \frac{10\sqrt{3} \cdot 5}{2} = 25\sqrt{3}$$

Vastaus $25\sqrt{3}$

609



Ratkaistaan x yhtälöstä

$$\sin 22{,}5^\circ = \frac{x}{2}$$

$$x = 2 \cdot \sin 22{,}5^\circ = 0{,}765...$$

$$x \approx 0{,}8$$

Ratkaistaan y yhtälöstä

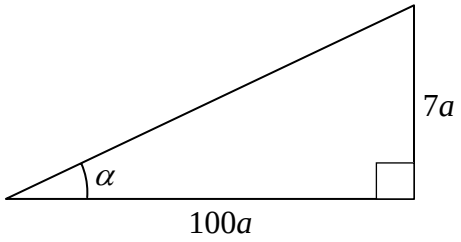
$$\cos 22{,}5^\circ = \frac{y}{2}$$

$$y = 2 \cdot \cos 22{,}5^\circ = 1{,}847...$$

$$y \approx 1{,}8 \text{ (km)}$$

Vastaus 0,8 km itään ja 1,8 km etelään.

610



Kun mäen kaltevuus on 7%, niin edettäessä 100 yksikköä vaakasuoraan noustaan 7 yksikköä ylöspäin. Ratkaistaan kaltevuuskulma α yhtälöstä

$$\tan \alpha = \frac{7a}{100a}$$

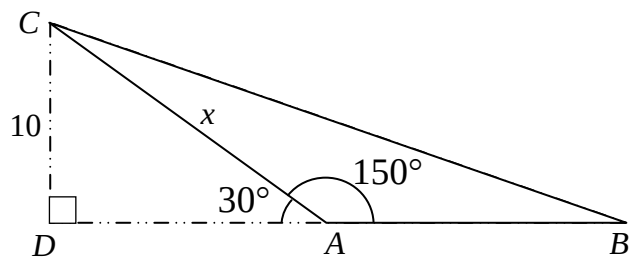
$$\tan \alpha = 0{,}07$$

$$\alpha = 4{,}004...^\circ$$

$$\alpha \approx 4^\circ$$

Vastaus Kaltevuuskulma on 4°

611



Kolmiosta *DAC* saadaan

$$\sin 30^\circ = \frac{10}{x}$$

$$x = \frac{10}{\sin 30^\circ}$$

$$x = \frac{10}{\frac{1}{2}}$$

$$x = 20$$

Vastaus $AC = 20$

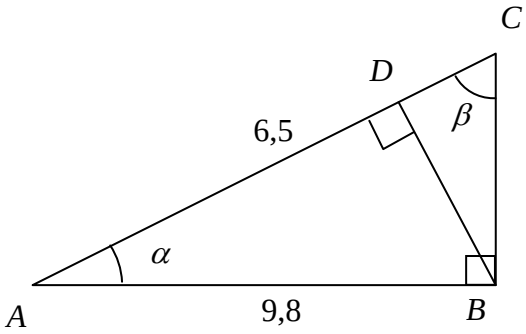
612

Ratkaistaan kulmat α ja β suorakulmaisesta kolmiosta *ABD*.

$$\cos\alpha = \frac{6,5}{9,8}$$

$$\alpha = \cos^{-1}\left(\frac{6,5}{9,8}\right)$$

$$\alpha = 48,450...^\circ \approx 48,5^\circ$$



Kulmaksi β saadaan

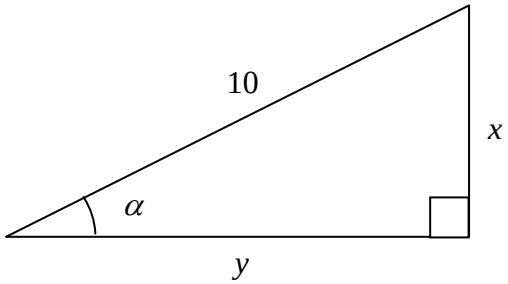
$$\beta = 90^\circ - \alpha = 41,549...^\circ \approx 41,5^\circ$$

Vastaus: 48,5° ja 41,5°

613

a) Ratkaistaan ensin kateetti *x*.

$$\sin\alpha = \frac{x}{10}$$



Toisaalta

$$\sin\alpha = \frac{1}{\sqrt{5}}$$

joten saadaan yhtälö

$$\frac{x}{10} = \frac{1}{\sqrt{5}}$$

$$x\sqrt{5} = 10$$

$$x = \overset{\sqrt{5})}{\frac{10}{\sqrt{5}}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}$$

Ratkaistaan kateetti *y*.

$$x^2 + y^2 = 10^2$$

$$y^2 = 100 - \left(2\sqrt{5}\right)^2$$

$$y^2 = 100 - 4 \cdot 5$$

$$y^2 = 80$$

$$y = \overset{(\pm)}{} \sqrt{80}$$

$$y = 4\sqrt{5}$$

Kateetit ovat $2\sqrt{5}$ ja $4\sqrt{5}$.

b)

$$\cos\alpha = \frac{y}{10}$$

$$\cos\alpha = \frac{4\sqrt{5}}{10}$$

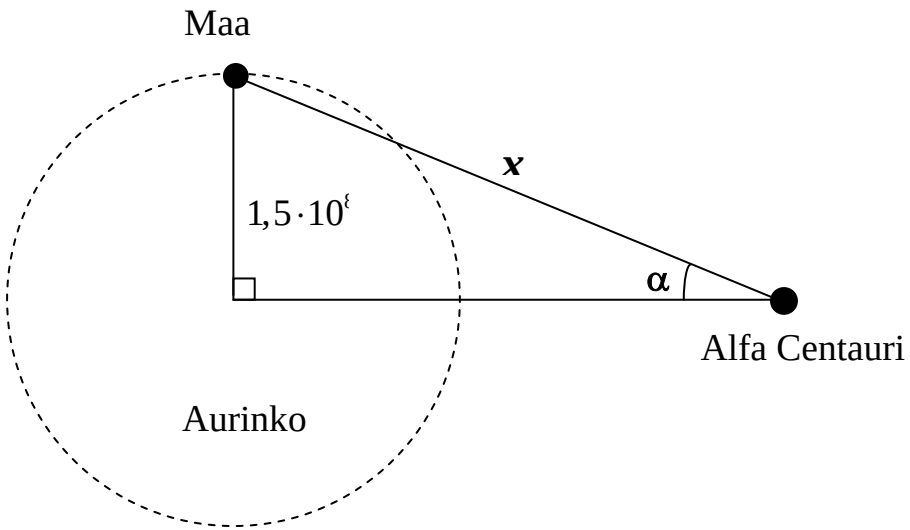
$$\cos\alpha = \frac{2\sqrt{5}}{5}$$

Vastaus: a) $2\sqrt{5}$ ja $4\sqrt{5}$ b) $\frac{2\sqrt{5}}{5}$

614

a) Kulmasekunti on $\frac{1}{3600}°$, joten $\alpha = 0{,}75 \cdot \frac{1}{3600}°$.

Saadaan yhtälö



$$\sin\alpha = \frac{1{,}5 \cdot 10^8}{x}$$

$$x\sin\alpha = 1{,}5 \cdot 10^8$$

$$x = \frac{1{,}5 \cdot 10^8}{\sin\alpha}$$

$$x = \frac{1{,}5 \cdot 10^8}{\sin\left(\frac{0{,}75}{3600}°\right)}$$

$$x = 4{,}125\ldots \cdot 10^{13} \approx 4{,}1 \cdot 10^{13}\ \text{(km)}$$

b) Kulmasekuntia $\frac{1}{3600}°$ vastaava etäisyys on

$$\sin\left(\frac{1}{3600}°\right) = \frac{1{,}5 \cdot 10^8}{y}$$

$$y = \frac{1{,}5 \cdot 10^8}{\sin\left(\frac{1}{3600}°\right)}$$

$$y = 3{,}093\ldots \cdot 10^{13}\ \text{(km)}$$

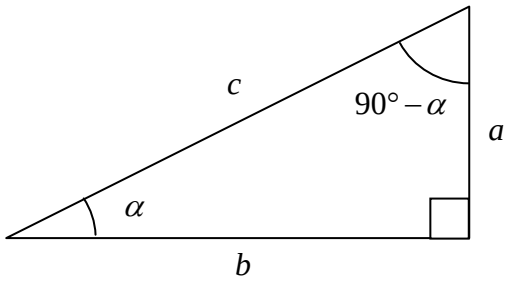
Tämä etäisyys vastaa siis yhtä parsekia (1 pc).

Siten Alfa Centaurin etäisyys Maasta parsekeina on

$\frac{x}{y} = 1{,}33\ldots \approx 1{,}3\ \text{(pc)}$.

Vastaus: a) $4{,}1 \cdot 10^{13}$ km b) 1,3 pc

615



Kun α on suorakulmaisen kolmion terävä kulma, niin toinen terävä kulma on $90^\circ - \alpha$. Merkitään kolmion sivuja kuvan mukaisesti.

a) $\sin\alpha = \frac{a}{c}$ ja $\cos(90^\circ - \alpha) = \frac{a}{c}$, joten

$\sin\alpha = \cos(90^\circ - \alpha)$

□

b) $\cos\alpha = \frac{b}{c}$ ja $\sin(90^\circ - \alpha) = \frac{b}{c}$, joten

$\cos\alpha = \sin(90^\circ - \alpha)$

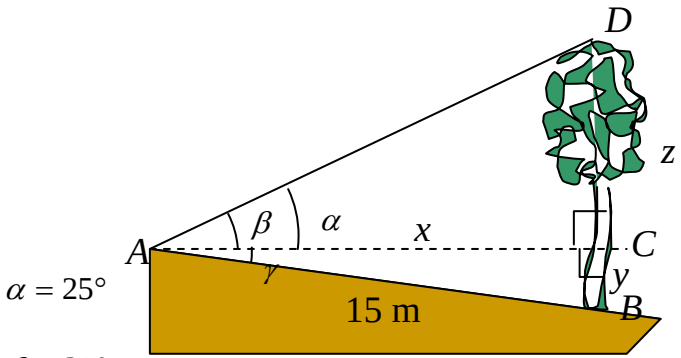
□

c) $\tan\alpha = \frac{a}{b}$ ja $\frac{\sin\alpha}{\cos\alpha} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{c} \cdot \frac{c}{b} = \frac{a}{b}$, joten

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

□

616



Suorakulmaisen kolmion *ABC* terävä kulma

$$\gamma = \beta - \alpha = 32^\circ - 25^\circ = 7^\circ.$$

Suorakulmaisesta kolmiosta *ABC* ratkaistaan *x* ja *y*.

$$\cos 7^\circ = \frac{x}{15}$$

$$x = 15\cos 7^\circ$$

$$x = 14{,}888...$$

$$\sin 7° = \frac{y}{15}$$

$$y = 15\sin 7°$$

$$y = 1{,}828...$$

Ratkaistaan z suorakulmaisesta kolmiosta ACD.

$$\tan\alpha = \frac{z}{x}$$

$$z = x\tan\alpha$$

$$z = 15\cos 7° \cdot \tan 25°$$

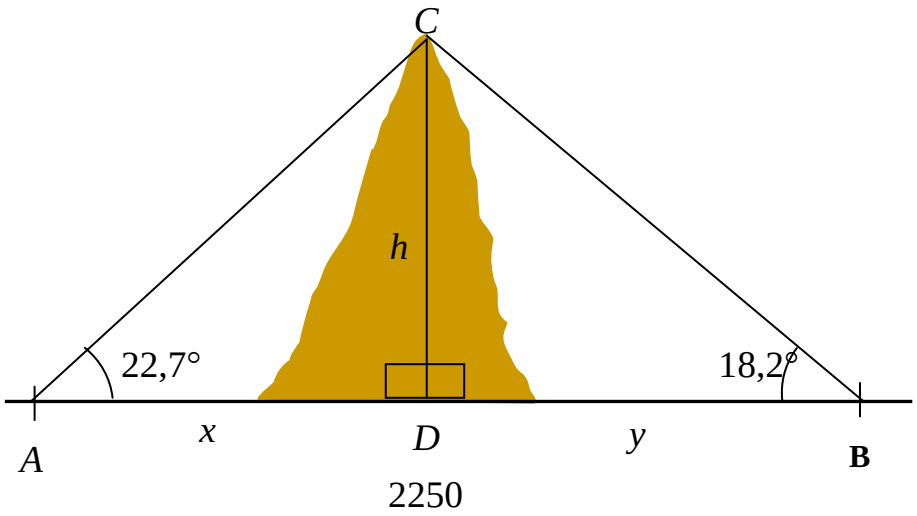
$$z = 6{,}942...$$

Puun korkeus on

$$y + z = 1{,}828... + 6{,}942... = 8{,}770... \approx 8{,}8 \text{ (m)}$$

Vastaus 8,8 m

617



Suorakulmaisesta kolmiosta ADC saadaan

$$\tan 22{,}7° = \frac{h}{x}$$

$$x = \frac{h}{\tan 22{,}7°}$$

Suorakulmaisesta kolmiosta DBC saadaan

$$\tan 18{,}2° = \frac{h}{y}$$

$$y = \frac{h}{\tan 18{,}2°}$$

Koska $x + y = 2250$, saadaan h ratkaistua yhtälöstä

$$\frac{h}{\tan 22,7°} + \frac{h}{\tan 18,2°} = 2250$$

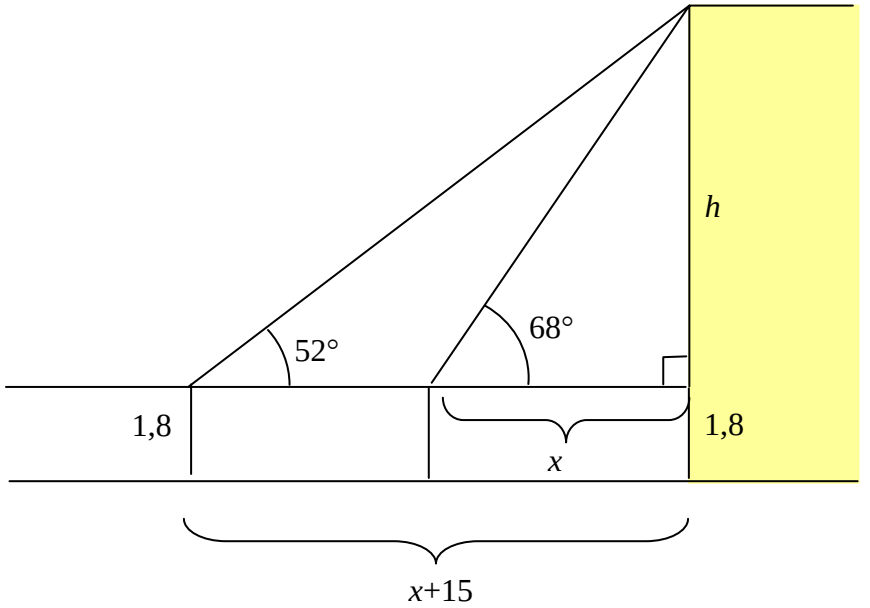
$$h\left(\frac{1}{\tan 22,7°} + \frac{1}{\tan 18,2°}\right) = 2250$$

$$h = \frac{2250}{\frac{1}{\tan 22,7°} + \frac{1}{\tan 18,2°}}$$

$$h = 414,20... \approx 414$$

Vastaus: 414 m korkeammalla

618



Suorakulmaisista kolmioista saadaan yhtälöt

$$\tan 68° = \frac{h}{x}$$

$$h = x \tan 68°$$

$$\tan 52° = \frac{h}{x+15}$$

$$h = (x+15) \tan 52°$$

Etäisyys x saadaan ratkaistua yhtälöstä

$$x\tan 68° = (x+15)\tan 52°$$

$$x\tan 68° = x\tan 52° + 15\tan 52°$$

$$x\tan 68° - x\tan 52° = 15\tan 52°$$

$$x(\tan 68° - \tan 52°) = 15\tan 52°$$

$$x = \frac{15\tan 52°}{\tan 68° - \tan 52°}$$

$$x = 16,064...$$

Saadaan

$$h = x\tan 68° = 16,064...\tan 68° = 39,760... \approx 40\,(\text{m})$$

Tornin korkeus on siten

$$h + 1,8 = 39,760... + 1,8 = 41,560... \approx 41,6$$

Vastaus Tornin korkeus on 41,6 m.

619

$AL = 10$ km

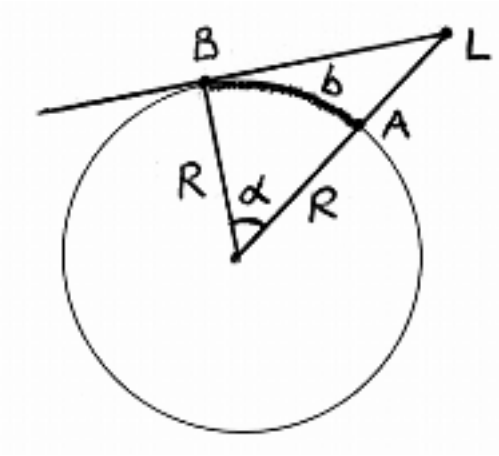
$R = 6400$ km

Lasketaan näkyvyysalue, eli kaaren AB pituus b.
Keskuskulma α saadaan yhtälöstä

$$\cos\alpha = \frac{R}{R+10}$$

$$\cos\alpha = \frac{6400}{6410}$$

$$\alpha = 3,200...^\circ \approx 3,2^\circ$$



Kaaren AB pituus on

$$b = \frac{\alpha}{360^\circ}\cdot 2\pi\cdot R$$

$$= \frac{3,2^\circ}{360^\circ}\cdot 2\pi\cdot 6400$$

$$= 357,538... < 400$$

joten Pariisi ei ole kaarella AB, eikä lentokoneesta voi nähdä Pariisia.

Vastaus Ei voi.

620

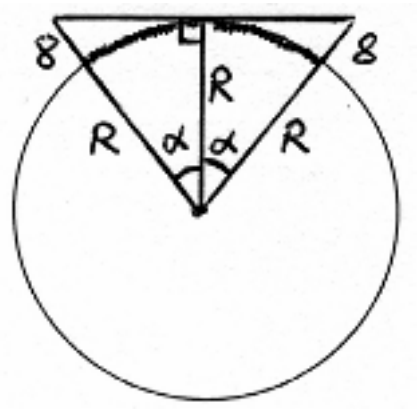
Lasketaan kulma α.

$$\cos\alpha = \frac{R}{R+8}$$

$$\cos\alpha = \frac{6400}{6400+0{,}008}$$

$$\alpha = 0{,}0905...^\circ \text{ siten}$$

$$2\alpha = 0{,}1811...^\circ$$

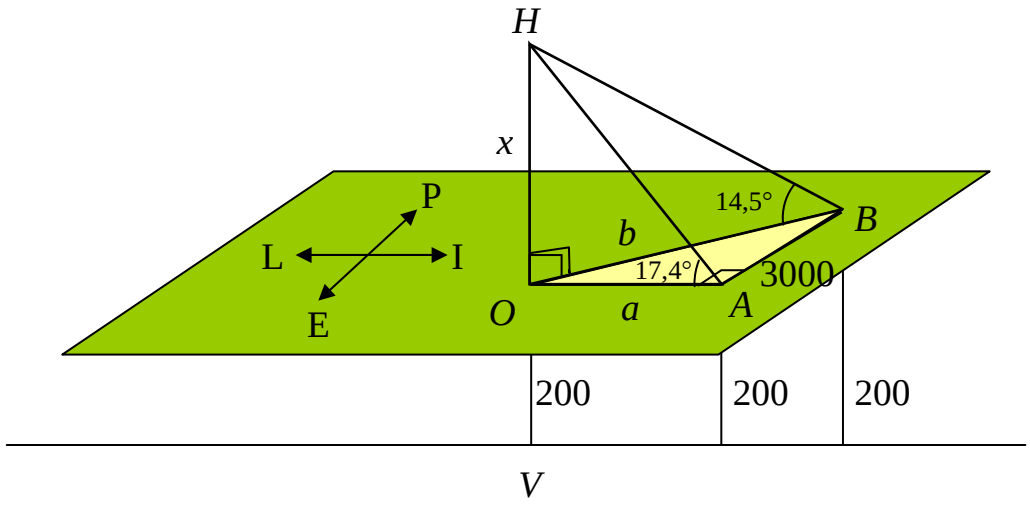


Kaaren pituus on

$$s = \frac{2\alpha}{360^\circ} \cdot 2\pi \cdot 6400 = 20{,}23... \approx 20 \text{ (km)}$$

Vastaus 20 km etäisyydellä toisistaan maanpintaa pitkin mitattuna

621

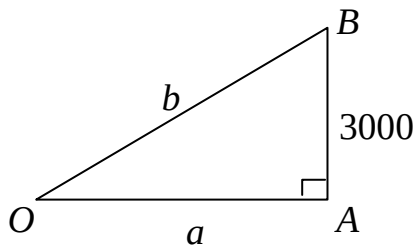


$$\tan 14{,}5^\circ = \frac{x}{b} \quad |_{\triangle OBH} \qquad \tan 17{,}4^\circ = \frac{x}{a} \quad |_{\triangle OAH}$$

$$b\tan 14{,}5^\circ = x \qquad a\tan 17{,}4^\circ = x$$

$$b = \frac{x}{\tan 14{,}5^\circ} \qquad a = \frac{x}{\tan 17{,}4^\circ}$$

$$b \approx 3{,}8667x \qquad a \approx 3{,}1910x$$



$$b^2 = a^2 + 3000^2$$

$$3{,}8667^2 x^2 = 3{,}1910^2 x^2 + 3000^2$$

$$\left(3{,}8667^2 - 3{,}1910^2\right) x^2 = 3000^2$$

$$x^2 = \frac{3000^2}{3{,}8667^2 - 3{,}1910^2}$$

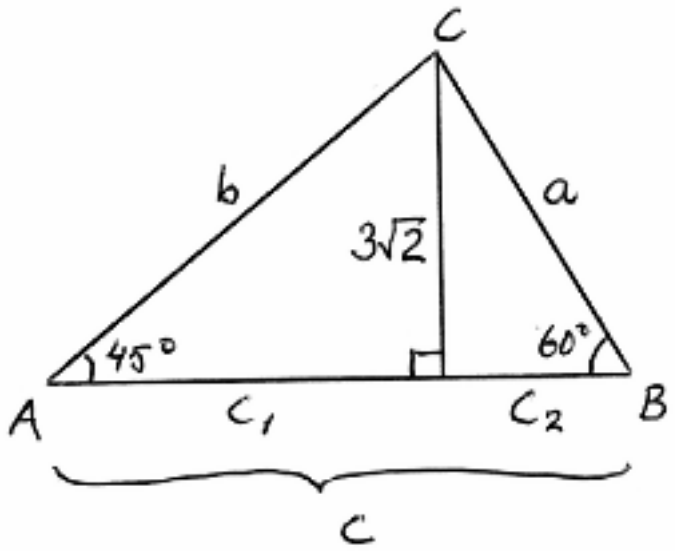
$$x = {}_{(}\pm_{)}\sqrt{\frac{3000^2}{3{,}8667^2 - 3{,}1910^2}}$$

$$x \approx 1373{,}77 \text{ (m)}$$

$$HV = x + 200 \approx 1573{,}77 \text{ (m)}$$

Vastaus 1570 m

622



$$\sin 45^\circ = \frac{3\sqrt{2}}{b}$$

$$b = \frac{3\sqrt{2}}{\sin 45^\circ} = \frac{3\sqrt{2}}{\frac{1}{\sqrt{2}}} = 3\sqrt{2} \cdot \sqrt{2} = 6$$

$$\sin 60^\circ = \frac{3\sqrt{2}}{a}$$

$$a = \frac{3\sqrt{2}}{\sin 60^\circ} = \frac{3\sqrt{2}}{\frac{\sqrt{3}}{2}} = \frac{6\sqrt{2}}{\sqrt{3}} = \frac{6\sqrt{6}}{3} = 2\sqrt{6}$$

$$\tan 45^\circ = \frac{3\sqrt{2}}{c_1}$$

$$c_1 = \frac{3\sqrt{2}}{\tan 45^\circ} = \frac{3\sqrt{2}}{1} = 3\sqrt{2}$$

$$\tan 60^\circ = \frac{3\sqrt{2}}{c_2}$$

$$c_2 = \frac{3\sqrt{2}}{\tan 60^\circ} = \overset{\sqrt{3})}{} \frac{3\sqrt{2}}{\sqrt{3}} = \frac{3\sqrt{3}\sqrt{2}}{3} = \sqrt{6}$$

Siis $c = c_1 + c_2 = 3\sqrt{2} + \sqrt{6}$

Vastaus $AC = 6,\ BC = 2\sqrt{6},\ AB = 3\sqrt{2} + \sqrt{6}$

623

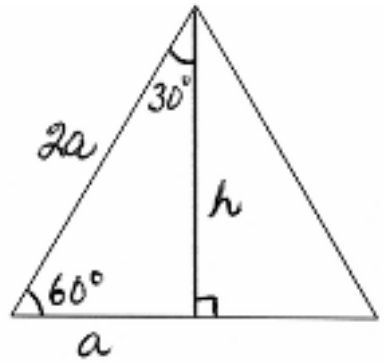
Piirretään tasasivuinen kolmio ja siihen korkeusjana.
Pythagoraan lauseen mukaan

$$h^2 + a^2 = (2a)^2$$

$$h^2 = 4a^2 - a^2$$

$$h^2 = 3a^2$$

$$h = \underset{(}{\pm}) a\sqrt{3}$$



a)

$$\sin 30^\circ = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{h}{2a} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{a}{h} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Vastaus $\sin 30^\circ = \frac{1}{2},\ \cos 30^\circ = \frac{\sqrt{3}}{2},\ \tan 30^\circ = \frac{1}{\sqrt{3}}$

b)

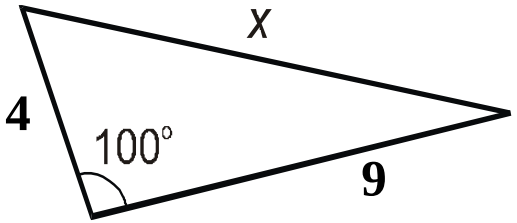
$$\sin 60^\circ = \frac{h}{2a} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{h}{a} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

Vastaus $\sin 60^\circ = \frac{\sqrt{3}}{2},\ \cos 60^\circ = \frac{1}{2},\ \tan 60^\circ = \sqrt{3}$

624



Ratkaistaan x kosinilauseella.

$$x^2 = 4^2 + 9^2 - 2 \cdot 4 \cdot 9 \cdot \cos 100^\circ$$

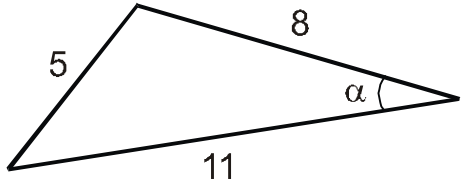
$$x^2 = 109{,}502...$$

$$x = {}_{(}\pm_{)}\sqrt{109{,}502...}$$

$$x = 10{,}464... \approx 10$$

Vastaus $x \approx 10$

625



Kosinilauseen mukaan

$$5^2 = 8^2 + 11^2 - 2 \cdot 8 \cdot 11 \cdot \cos\alpha$$

$$25 = 64 + 121 - 176\cos\alpha$$

$$176\cos\alpha = 160$$

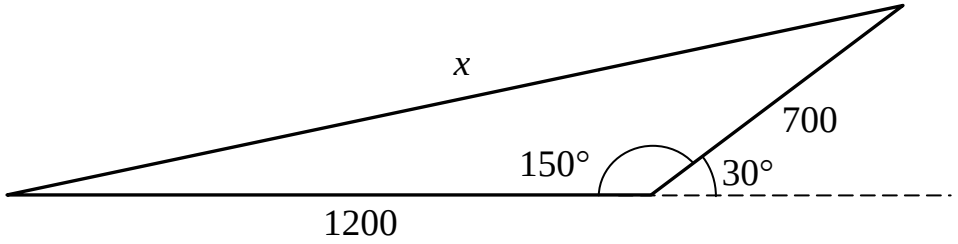
$$\cos\alpha = \frac{160}{176}$$

$$\alpha = \cos^{-1}\left(\frac{160}{176}\right)$$

$$\alpha = 24{,}61997...^\circ \approx 24{,}6^\circ$$

Vastaus $\alpha \approx 24{,}6^\circ$

626



Kosinilauseella saadaan

$$x^2 = 1200^2 + 700^2 - 2 \cdot 1200 \cdot 700 \cdot \cos 150^\circ$$

$$x^2 = 3{,}3849... \cdot 10^6$$

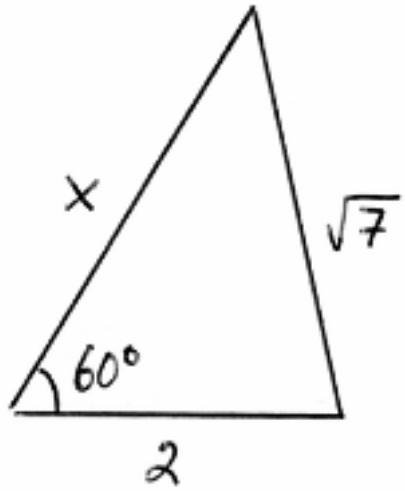
$$x = {}_{(}\pm_{)}\sqrt{3{,}3849... \cdot 10^6}$$

$$x = 1839{,}815...$$

$$x \approx 1800$$

Vastaus 1800 m

627



Kosinilauseella saadaan

$$\left(\sqrt{7}\right)^2 = 2^2 + x^2 - 2 \cdot 2 \cdot x \cdot \cos 60^\circ$$

$$7 = 4 + x^2 - 4x \cdot \frac{1}{2}$$

$$x^2 - 2x - 3 = 0$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1}$$

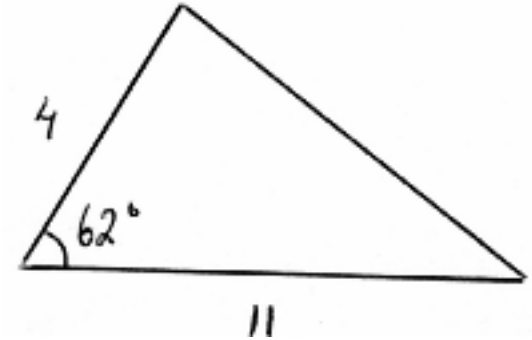
$$x = \frac{2 \pm 4}{2}$$

$x = 3$ tai $x = -1$

(kelpaa) (ei kelpaa)

Vastaus 3

628



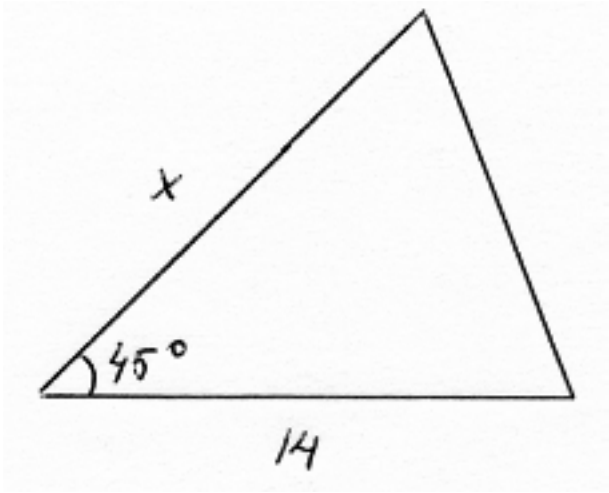
Kolmion ala on

$$A = \frac{1}{2} \cdot 4 \cdot 11 \cdot \sin 62^\circ$$

$$= 19,4248... \approx 19,4$$

Vastaus $19,4$

629



Pinta-ala on 56, joten saadaan yhtälö

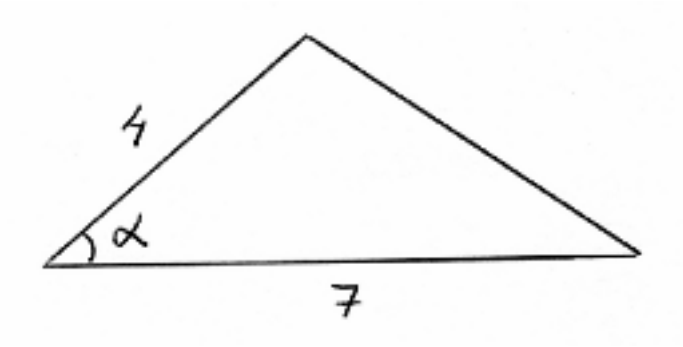
$$56 = \frac{1}{2} \cdot 14 \cdot x \cdot \sin 45^\circ$$

$$56 = 7x \cdot \frac{1}{\sqrt{2}}$$

$$x = \frac{56\sqrt{2}}{7} = 8\sqrt{2}$$

Vastaus $8\sqrt{2}$

630



Kolmion ala on 8, joten saadaan yhtälö

$$\frac{1}{2}\cdot 4\cdot 7\cdot \sin\alpha = 8$$

$$14\sin\alpha = 8$$

$$\sin\alpha = \frac{8}{14}$$

$$\alpha = \sin^{-1}\left(\frac{8}{14}\right)$$

$$\alpha = 34{,}849...^\circ$$

$$\alpha \approx 35^\circ$$

Suplementtikulmien sinit ovat yhtä suuret, joten kulmaksi α kelpaa myös

$$180^\circ - 34{,}849...^\circ = 145{,}150...^\circ \approx 145^\circ$$

Vastaus 35° tai 145°

631

Sinilauseen mukaan

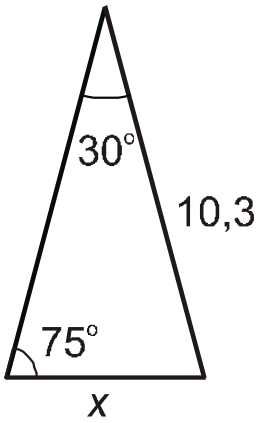
$$\frac{x}{\sin 30^\circ} = \frac{10{,}3}{\sin 75^\circ}$$

$$x\cdot \sin 75^\circ = 10{,}3\cdot \sin 30^\circ$$

$$x = \frac{10{,}3\cdot \sin 30^\circ}{\sin 75^\circ}$$

$$x = 5{,}3316...$$

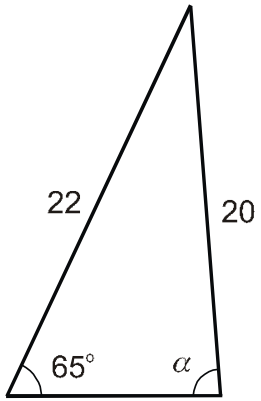
$$x \approx 5{,}3$$



Vastaus $x \approx 5{,}3$

632

Sinilauseen mukaan



$$\frac{22}{\sin\alpha} = \frac{20}{\sin 65^\circ}$$

$$20 \cdot \sin\alpha = 22 \cdot \sin 65^\circ$$

$$\sin\alpha = \frac{22 \cdot \sin 65^\circ}{20}$$

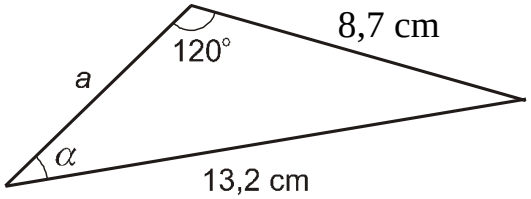
$$\sin\alpha = 0{,}9969...$$

$\alpha = 85{,}515...^\circ$ tai $\alpha = 180 - 85{,}515...^\circ$

$\alpha \approx 86^\circ$ tai $\alpha \approx 94^\circ$

Vastaus $\alpha \approx 86^\circ$ tai $\alpha \approx 94^\circ$

633



a) Sinilauseen mukaan

$$\frac{8{,}7}{\sin\alpha} = \frac{13{,}2}{\sin 120^\circ}$$

$$13{,}2 \cdot \sin\alpha = 8{,}7 \cdot \sin 120^\circ$$

$$\sin\alpha = 0{,}570789...^\circ$$

$\alpha = 34{,}8052...^\circ$ tai $\alpha = 180^\circ - 34{,}8052...^\circ$

$\alpha = 145{,}1947...^\circ$

Koska $34{,}8052...^\circ + 120^\circ < 180^\circ$ ja $145{,}1947...^\circ + 120^\circ > 180^\circ$, niin vain $\alpha = 34{,}8052...^\circ$ kelpaa.

Siis $\alpha \approx 35^\circ$

b) Sivu *a* voidaan ratkaista joko a-kohdan tulosta hyödyntäen käyttäen joko sini- tai kosinilausetta (tapa1) tai sitten ilman a-kohdassa laskettua kulmaa käyttäen kosinilausetta (tapa 2).

Tapa1 Sivun a vastainen kulma β on

$$= 180^\circ - \alpha - 120^\circ = 25{,}1947...^\circ$$

Sinilauseen mukaan

$$\frac{8{,}7}{\sin 34{,}8052...^\circ} = \frac{a}{\sin 25{,}1947...^\circ}$$

$$a \cdot \sin 34{,}8052...^\circ = 8{,}7 \cdot \sin 25{,}1947...^\circ$$

$$a = 6{,}48847...$$

$$a \approx 6{,}5 \ (\text{cm})$$

Tapa 2 Kosinilauseen mukaan

$$13{,}2^2 = a^2 + 8{,}7^2 - 2 \cdot a \cdot 8{,}7 \cdot \cos 120^\circ$$

$$a^2 + 8{,}7a - 98{,}55 = 0$$

$$a = \frac{-8{,}7 \pm \sqrt{8{,}7^2 - 4 \cdot 1(-98{,}55)}}{2 \cdot 1}$$

$$a = \frac{-8{,}7 \pm \sqrt{469{,}89}}{2}$$

$a = -15{,}188...$ tai $a = 6{,}488... \approx 6{,}5$

(ei kelpaa) (kelpaa)

Vastaus a) $\alpha \approx 35^\circ$ b) $a \approx 6{,}5$ cm

634

Kulma $\gamma = 180^\circ - 72^\circ - 49^\circ = 59^\circ$

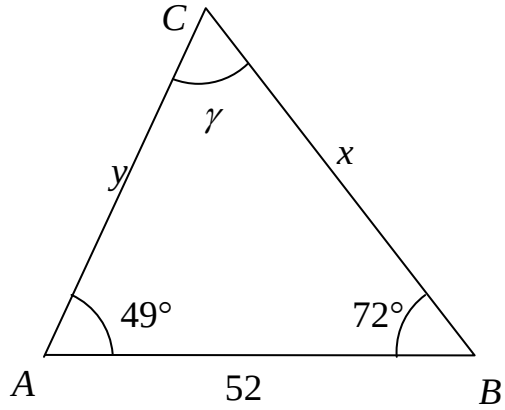
Ratkaistaan sivut x ja y sinilauseella.

$$\frac{52}{\sin 59^\circ} = \frac{x}{\sin 49^\circ}$$

$$x \sin 59^\circ = 52 \sin 49^\circ$$

$$x = \frac{52 \sin 49^\circ}{\sin 59^\circ}$$

$$x = 45{,}784... \approx 46 \ (\text{m})$$



$$\frac{52}{\sin 59^\circ} = \frac{y}{\sin 72^\circ}$$

$$y \sin 59^\circ = 52 \sin 72^\circ$$

$$y = \frac{52 \sin 72^\circ}{\sin 59^\circ}$$

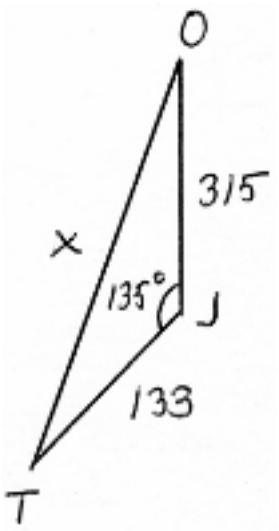
$$y = 57{,}6957... \approx 58 \ (\text{m})$$

Pinta-ala on

$$A = \frac{1}{2} \cdot 52 \cdot 57{,}6957... \sin 49^\circ = 1132{,}132... \approx 1100 \ (\text{m}^2)$$

Vastaus: $BC \approx 46$ m, $AC \approx 58$ m ja ala on 11 aaria

635



Kosinilauseella saadaan

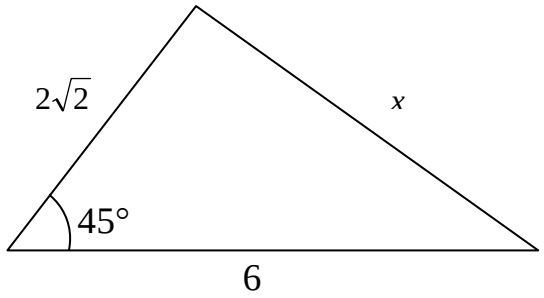
$$x^2 = 133^2 + 315^2 - 2 \cdot 133 \cdot 315 \cdot \cos 135^\circ$$

$$x^2 = 176\,162,47...$$

$$x = 419,7... \approx 420 \text{ (km)}$$

Vastaus 420 km

636



Kosinilauseella saadaan

$$x^2 = 6^2 + (2\sqrt{2})^2 - 2 \cdot 6 \cdot 2\sqrt{2} \cos 45^\circ$$

$$x^2 = 36 + 4 \cdot 2 - 24\sqrt{2} \cdot \frac{1}{\sqrt{2}}$$

$$x^2 = 44 - 24$$

$$x^2 = 20$$

$$x = (\pm)\sqrt{20}$$

$$x = 2\sqrt{5}$$

Pinta-ala on

$$A = \frac{1}{2} \cdot 6 \cdot 2\sqrt{2} \cdot \sin 45^\circ = 6\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 6$$

Vastaus: $2\sqrt{5}$, 6

637

Lasketaan suurin kulma

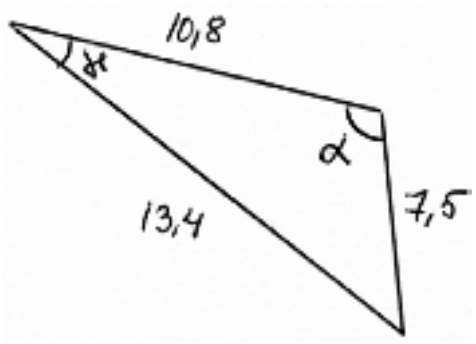
$$13,4^2 = 7,5^2 + 10,8^2 - 2 \cdot 7,5 \cdot 10,8 \cdot \cos\alpha$$

$$179,56 = 56,25 + 116,64 - 162\cos\alpha$$

$$162\cos\alpha = -6,67$$

$$\cos\alpha = -\frac{6,67}{162}$$

$$\alpha \approx 92,35...^\circ \approx 92^\circ$$



Lasketaan pienin kulma

$$7,5^2 = 13,4^2 + 10,8^2 - 2 \cdot 13,4 \cdot 10,8 \cdot \cos\gamma$$

$$289,44\cos\gamma = 239,95$$

$$\cos\gamma = \frac{239,95}{289,44}$$

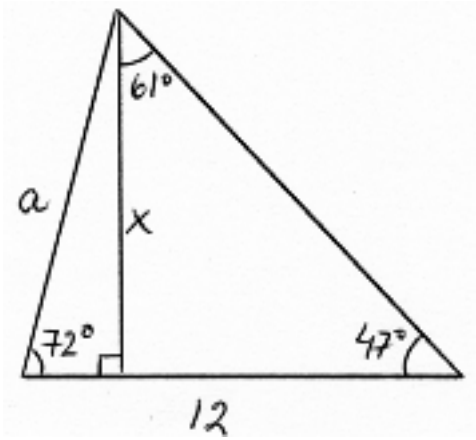
$$\gamma = 34,00... \approx 34^\circ$$

Kolmas kulma on

$$180^\circ - 92,35...^\circ - 34,00...^\circ = 53,64...^\circ \approx 54^\circ$$

Vastaus Kolmion kulmat ovat 34°, 54° ja 92°

638



Tapa 1

$$\frac{12}{\sin 61^\circ} = \frac{a}{\sin 47^\circ} \qquad |\text{sinilause}$$

$$a\sin 61^\circ = 12\sin 47^\circ$$

$$a = \frac{12\sin 47^\circ}{\sin 61^\circ} = 10,034...$$

$$\sin 72^\circ = \frac{x}{a} \qquad |\cdot a$$

$$x = a\sin 72^\circ$$

$$x = \frac{12\sin 47^\circ}{\sin 61^\circ} \cdot \sin 72^\circ \approx 9,5 \text{ (m)}$$

Vastaus 9,5 m

Tapa 2

$$\tan 72^\circ = \frac{x}{a} \qquad |\cdot a$$

$$a\tan 72^\circ = x \qquad |: \tan 72^\circ$$

$$a = \frac{x}{\tan 72^\circ}$$

$$\tan 47^\circ = \frac{x}{b} \qquad |\cdot b$$

$$b\tan 47^\circ = x \qquad |: \tan 47^\circ$$

$$b = \frac{x}{\tan 47^\circ}$$

Koska $a + b = 12$, saadaan yhtälö

$$\frac{x}{\tan 72^\circ} + \frac{x}{\tan 47^\circ} = 12 \qquad |\cdot \tan 72^\circ \cdot \tan 47^\circ$$

$$x\tan 47^\circ + x\tan 72^\circ = 12\tan 72^\circ \tan 47^\circ$$

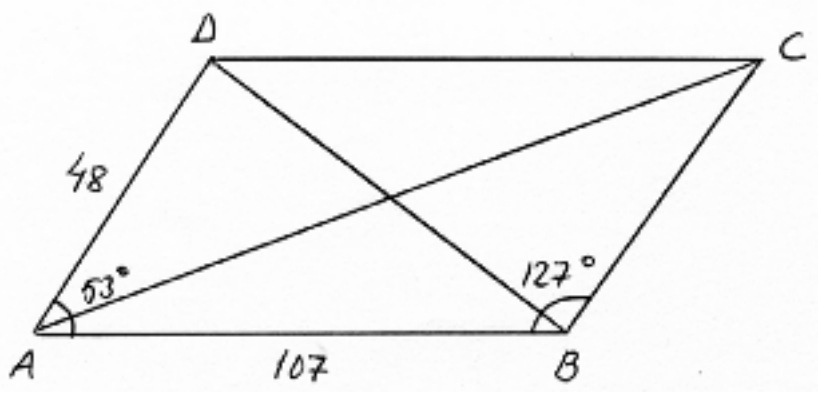
$$x(\tan 47^\circ + \tan 72^\circ) = 12\tan 72^\circ \tan 47^\circ$$

$$x = \frac{12\tan 72^\circ \tan 47^\circ}{\tan 47^\circ + \tan 72^\circ}$$

$$x \approx 9{,}5 \text{ (m)}$$

Vastaus 9,5 m

639



Kolmiosta *ABD* saadaan lävistäjä *BD* kosinilauseella

$$BD^2 = 107^2 + 48^2 - 2\cdot 107\cdot 48\cdot \cos 53^\circ$$

$$BD^2 = 7571{,}156...$$

$$BD = {}_{(-)}\sqrt{7571{,}156...}$$

$$BD = 87{,}012... \approx 87 \text{ (m)}$$

Lävistäjä *AC* saadaan kolmiosta *ABC* kosinilauseella

$$AC^2 = 107^2 + 48^2 - 2\cdot 107\cdot 48\cdot \cos 127^\circ$$

$$AC^2 = 19934{,}84...(\text{m}^2)$$

$$AC = {}_{(-)}\sqrt{19934{,}84...}$$

$$AC = 141{,}19... \approx 141 \text{ (m)}$$

Suunnikkaan ala on kaksi kertaa kolmion *ABD* ala

$$A = 2 \cdot \frac{1}{2} \cdot 107 \cdot 48 \cdot \sin 53^\circ$$

$$= 4101,79...$$

$$A = 4101,79... \text{ m}^2 \approx 4100 \text{ m}^2 = 0,41 \text{ ha}$$

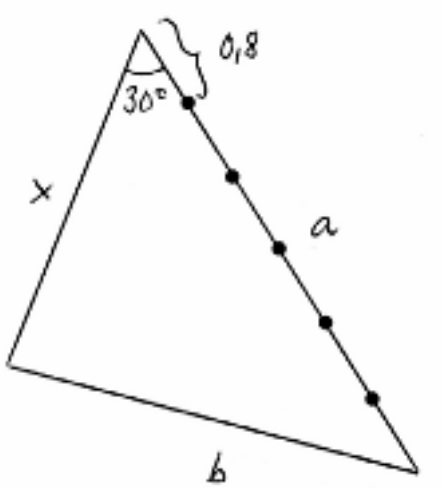
Vastaus Lävistäjät ovat 87 m ja 141 m.
Pinta-ala on 0,41 ha.

640

Yksikkönä on senttimetri.

$a = 6 \cdot 0,8 = 4,8$

$b = 4,6$



Kosinilauseella saadaan

$$b^2 = x^2 + a^2 - 2xa\cos 30^\circ$$

$$4,6^2 = x^2 + 4,8^2 - 2 \cdot 4,8x \cdot \frac{\sqrt{3}}{2}$$

$$4,6^2 = x^2 + 4,8^2 - 4,8\sqrt{3}x$$

$$x^2 - 4,8\sqrt{3}x + 1,88 = 0$$

$$x = \frac{4,8\sqrt{3} \pm \sqrt{\left(4,8\sqrt{3}\right)^2 - 4 \cdot 1 \cdot 1,88}}{2 \cdot 1}$$

$$x = \frac{2 \cdot 2,4\sqrt{3} \pm \sqrt{\left(2 \cdot 2,4\sqrt{3}\right)^2 - 4 \cdot 1 \cdot 1,88}}{2}$$

$$x = \frac{2 \cdot 2,4\sqrt{3} \pm 2\sqrt{2,4^2 \cdot 3 - 1,88}}{2}$$

$$x = 2,4\sqrt{3} \pm \sqrt{2,4^2 \cdot 3 - 1,88}$$

$x_1 = 2,4\sqrt{3} - 2\sqrt{2,4^2 \cdot 3 - 1,88} = 0,232... < 0,8$ ei kelpaa

$x_2 = 2,4\sqrt{3} + 2\sqrt{2,4^2 \cdot 3 - 1,88} = 8,081$ kelpaa

Välien määrä on $\frac{x_2}{0,8} = 10,10... \approx 10$

Vasemmalla puolella on yhteensä 11 kurkea, joten johtajakurjen jälkeen on 10 kurkea.

Vastaus 10 kurkea.

641

Kolmion pinta-ala on $A = \frac{1}{2}ab\sin\gamma$.

Huippukulman γ laskemiseksi saadaan yhtälö

$$\frac{1}{2}\cdot 8\cdot 8\sin\gamma = 16$$

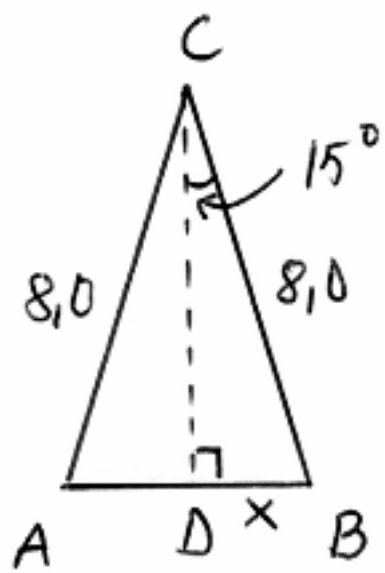
$$\sin\gamma = \frac{1}{2}$$

$$\gamma = 30^\circ \text{ tai } \gamma = 150^\circ \quad \left| \begin{array}{l} \text{Suplementtikulmien} \\ \text{sinit ovat yhtä suuret} \end{array} \right.$$

1) Jos kulma $\gamma = 30^\circ$, niin

$$\sphericalangle BCD = \frac{30^\circ}{2} = 15^\circ$$

$$\sphericalangle DBC = 90^\circ - 15^\circ = 75^\circ$$



Merkitään $x = DB$.

$$\frac{x}{8} = \cos 75^\circ$$

$$x = 8\cos 75^\circ = 2{,}070...$$

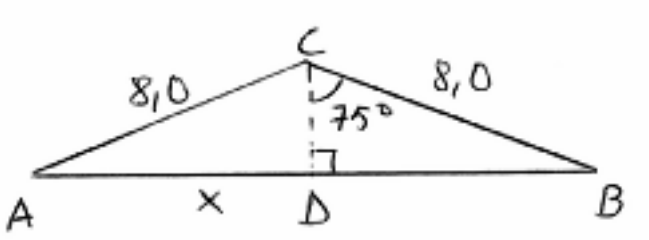
$$AB = 2x = 4{,}141...$$

$$AB \approx 4{,}1 \text{ (cm)}$$

2) Jos kulma $\gamma = 150^\circ$, niin

$$\sphericalangle DCA = \frac{50^\circ}{2} = 75^\circ$$

$$\sphericalangle CAD = 90^\circ - 75^\circ = 15^\circ$$



Merkitään $x = AD$.

$$\frac{x}{8} = \cos 15^\circ$$

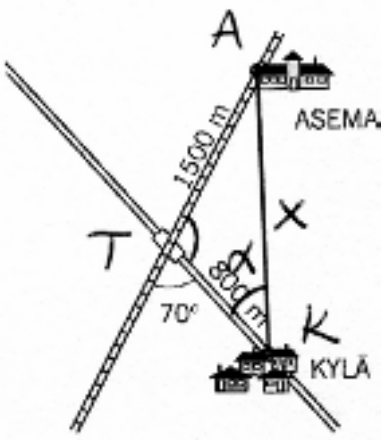
$$x = 8\cos 15^\circ = 7{,}727...$$

$$AB = 2x = 15{,}454...$$

$$AB \approx 15{,}5 \text{ (cm)}$$

Vastaus Huippukulma on 30° ja kanta on 4,1 cm tai huippukulma on 150° ja kanta 15,5 cm.

642



$TK = 800\ \text{m} = 0{,}8\ \text{km}$

$TA = 1500\ \text{m} = 1{,}5\ \text{km}$

$\sphericalangle ATK = 180^\circ - 70^\circ = 110^\circ$

Kosinilause

$x^2 = 0{,}8^2 + 1{,}5^2 - 2 \cdot 0{,}8 \cdot 1{,}5 \cos 110^\circ$

$x^2 = 3{,}710...$

$x = (\pm) 1{,}926...\ (\text{km})$

Sinilause

$$\frac{x}{\sin 110^\circ} = \frac{1{,}5}{\sin \alpha}$$

$$\sin \alpha = \frac{1{,}5 \cdot \sin 110^\circ}{1{,}926...} = 0{,}731...$$

$\alpha = 47{,}03... \approx 47^\circ$ kelpaa

tai $\alpha = 180^\circ - 47{,}03...^\circ = 132{,}96...^\circ \approx 133^\circ$

133° ei kelpaa, sillä $110^\circ + 133 > 180^\circ$

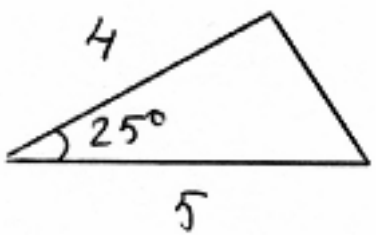
Vastaus 47°

643

Kolmion ala on

$$A = \frac{1}{2} \cdot 4 \cdot 5 \cdot \sin 25^\circ$$

$= 10 \sin 25^\circ = 4{,}2261...$



Määritetään kulma α niin, että ala on $1{,}2A$.

$$\frac{1}{2} \cdot 4 \cdot 5 \cdot \sin \alpha = 1{,}2 \cdot 10 \sin 25^\circ$$

$$\sin \alpha = 1{,}2 \sin 25^\circ$$

$$\sin \alpha = 0{,}5071...$$

$\alpha = 30{,}473...$ tai $\alpha = 149{,}526...^\circ$

Molemmat kulmat kelpaavat, sillä $\alpha + 25^\circ < 180^\circ$.

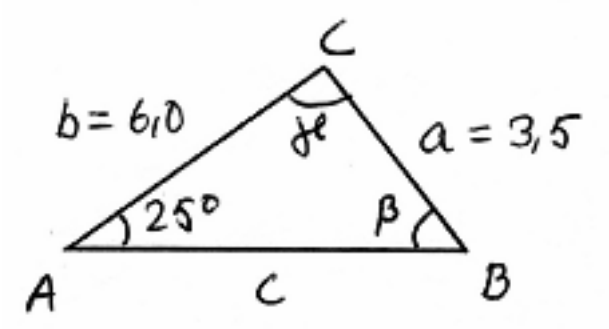
Kulmaa on siis kasvatettava

$30{,}473...^\circ - 25^\circ = 5{,}473...^\circ \approx 5^\circ$

$149{,}526...^\circ - 25^\circ = 124{,}526...^\circ \approx 125^\circ$

Vastaus 5° tai 125°

644



$$\frac{3,5}{\sin 25^\circ} = \frac{6,0}{\sin\beta} \qquad \text{sinilause}$$

$$3,5 \cdot \sin\beta = 6,0 \cdot \sin 25^\circ$$

$$\sin\beta = \frac{6,0 \cdot \sin 25^\circ}{3,5} \qquad \text{suplementtikulmat!}$$

$$\beta = 46,42...^\circ \quad \text{tai} \quad \beta = 180^\circ - 46,42...^\circ = 133,57...^\circ$$

$$\gamma = 108,57...^\circ \quad \text{tai} \quad \gamma = 21,42...^\circ$$

Sivu c voidaan laskea joko sini- tai kosinilauseella:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$c = {}_{(\pm)}\sqrt{a^2 + b^2 - 2ab\cos\gamma}$$

$$c = {}_{(\pm)}\sqrt{3,5^2 + 6,0^2 - 2 \cdot 3,5 \cdot 6,0\cos\gamma} \qquad \left| \begin{array}{l} \gamma = 108,57...^\circ \\ \text{tai} \\ \gamma = 21,42...^\circ \end{array} \right.$$

$$c = 7,85... \text{ (cm)} \quad \text{tai} \quad c = 3,02... \text{ (cm)}$$

Kolmion ala saadaan kaavalla

$$A = \frac{1}{2}bc\sin\alpha$$

$$A = \frac{1}{2} \cdot 6,0 \cdot c \cdot \sin 25^\circ \qquad \left| \begin{array}{l} c = 7,85... \\ \text{tai} \\ c = 3,02... \end{array} \right.$$

$$A = 9,95... \text{ (cm}^2\text{)} \quad \text{tai} \quad A = 3,83... \text{ (cm}^2\text{)}$$

Vastaus $\begin{cases} c = 7,9 \text{ cm} \\ \beta = 46^\circ \\ \gamma = 109^\circ \\ A = 10,0 \text{ cm}^2 \end{cases}$ tai $\begin{cases} c = 3,0 \text{ cm} \\ \beta = 134^\circ \\ \gamma = 21^\circ \\ A = 3,8 \text{ cm}^2 \end{cases}$

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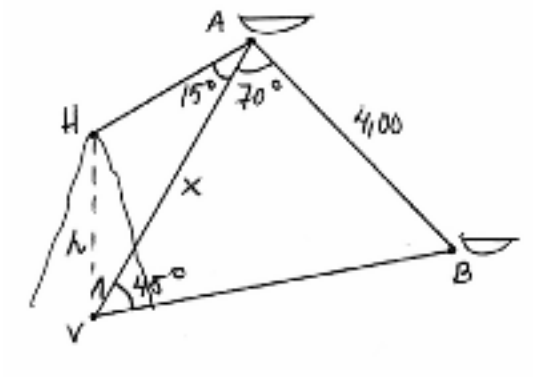
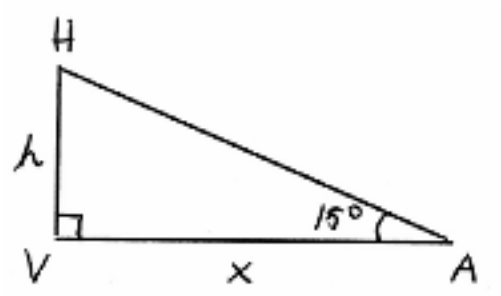
Vuorelta katsottuna piste *A* on pohjoisessa ja piste *B* koillisessa, joten kulma $AVB = 45^\circ$ ja edelleen kulma $VBA = 180^\circ - 70^\circ - 45^\circ = 65^\circ$. Sovelletaan sinilausetta kolmioon *VBA*.



$$\frac{x}{\sin 65^\circ} = \frac{4{,}00}{\sin 45^\circ}$$

$$x = \frac{4{,}00 \sin 65^\circ}{\sin 45^\circ}$$

Kolmiosta *VAH* saadaan



$$\frac{h}{x} = \tan 15^\circ$$

$$h = x \tan 15^\circ$$

$$h = \frac{4{,}00 \sin 65^\circ}{\sin 45^\circ} \tan 15^\circ$$

$$h = 1{,}3737...$$

$$h \approx 1{,}37 \text{ (km)}$$

Vastaus Huipun korkeus on 1 370 m

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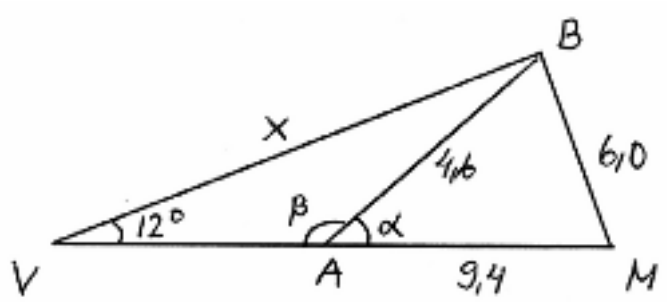

Kosinilause

$$6{,}0^2 = 9{,}4^2 + 4{,}6^2 - 2 \cdot 9{,}4 \cdot 4{,}6 \cos\alpha$$

$$2 \cdot 9{,}4 \cdot 4{,}6 \cos\alpha = 9{,}4^2 + 4{,}6^2 - 6{,}0^2$$

$$\cos\alpha = \frac{9{,}4^2 + 4{,}6^2 - 6{,}0^2}{2 \cdot 9{,}4 \cdot 4{,}6} = 0{,}850...$$

$$\alpha = 31{,}77...^\circ$$

Siten kulmaksi β saadaan

$$\beta = 180^\circ - 31{,}77...^\circ = 148{,}22...^\circ$$

Sinilause

$$\frac{x}{\sin\beta} = \frac{4{,}6}{\sin 12^\circ}$$

$$x = \frac{4{,}6 \cdot \sin 148{,}22...^\circ}{\sin 12^\circ} = 11{,}64... \approx 12 \text{ (km)}$$

Vastaus 12 km

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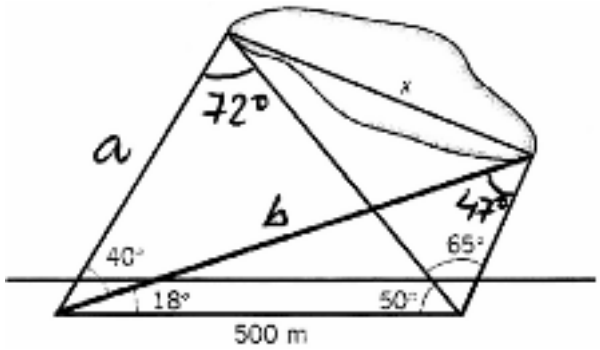

$x^2 = a^2 + b^2 - 2ab\cos 40^\circ$ |kosinilause

Lasketaan ensin a ja b.

$$\frac{a}{\sin 50^\circ} = \frac{500}{\sin 72^\circ} \quad \text{|sinilause}$$

$$a = 402,733...$$

$$\frac{b}{\sin 115^\circ} = \frac{500}{\sin 47^\circ} \quad \text{|sinilause}$$

$$b = 619,609...$$

Siis

$$x = {}_{(}\pm_{)}\sqrt{a^2 + b^2 - 2ab\cos 40^\circ} \quad \begin{vmatrix} a = 402,733... \\ b = 619,609... \end{vmatrix}$$

$$x = 404,7...$$

$$x \approx 400 \text{ (m)}$$

Vastaus 400 m

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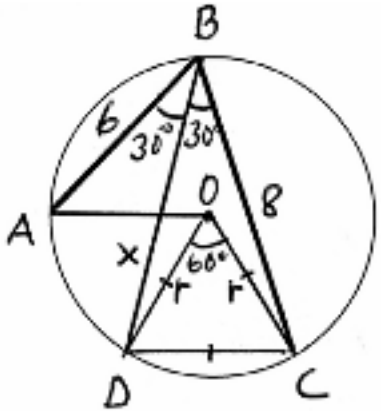
Olkoon r ympyrän säde.
30° kehäkulmaa vastaava keskuskulma on 60°.
Kolmiot AOD ja DOC ovat siten tasasivuisia ja
$AD = DC = r$.



Kolmioista ADB ja DCB saadaan yhtälöpari

$$\begin{vmatrix} r^2 = x^2 + 6^2 - 2 \cdot x \cdot 6 \cdot \cos 30^\circ & \cdot 1 \\ r^2 = x^2 + 8^2 - 2 \cdot x \cdot 8 \cdot \cos 30^\circ & \cdot(-1) \end{vmatrix}$$

$$\begin{cases} r^2 = x^2 + 6^2 - 2 \cdot x \cdot 6 \cdot \dfrac{\sqrt{3}}{2} \\ -r^2 = -x^2 - 8^2 + 2 \cdot x \cdot 8 \cdot \dfrac{\sqrt{3}}{2} \end{cases}$$

$$0 = -28 + x \cdot 2\sqrt{3}$$

$$x \cdot 2\sqrt{3} = 28 \quad |: 2\sqrt{3}$$

$$x = \frac{28}{2\sqrt{3}} = \frac{14}{\sqrt{3}} = \frac{14\sqrt{3}}{3}$$

Vastaus Jänne on $\dfrac{14\sqrt{3}}{3}$

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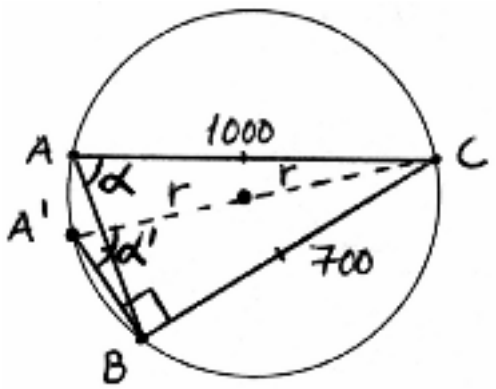

Kyseinen piste on kolmion keskinormaalien leikkauspiste.
Tämä piste on kolmion ympäri piirretyn ympyrän keskipiste.
Tehtävänä on siis määrittää ympyrän säde.

Ratkaistaan kulma α.

$$700^2 = 500^2 + 1000^2 - 2 \cdot 500 \cdot 1000 \cdot \cos\alpha$$

$$2 \cdot 500 \cdot 1000 \cos\alpha = 500^2 + 1000^2 - 700^2$$

$$\cos\alpha = \frac{500^2 + 1000^2 - 700^2}{2 \cdot 500 \cdot 1000} = 0,76$$

$$\alpha = 40,535...^\circ$$

Samaa kaarta vastaavat kehäkulmat ovat yhtä suuret. Siirretään kolmion kärkeä A niin, että $A'C$ on ympyrän halkaisija. Tällöin kulma $\alpha' = \alpha$ ja kulma $A'BC$ on puoliympyrää vastaavana kehäkulmana 90°.
Suorakulmaisesta kolmiosta $A'BC$ saadaan

$$\sin\alpha' = \frac{700}{2r}$$

$$2r\sin\alpha = 700$$

$$r = \frac{700}{2\sin\alpha}$$

$$r = \frac{700}{2\sin 40,535...^\circ}$$

$$r = 538,52...$$

$$r \approx 500 \text{ (m)}$$

Vastaus Kokoomarasti on 500 metrin päässä rasteista.